TRINITY COLLEGE

Semester One Examination, 2020

Question/Answer booklet

MATHEMATICS SPECIALIST UNIT 3

Section One: Calculator-free

SOLUTIONS

WA student number: In figures

In words ____________________

Your name ____________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Number of additional answer booklets used (if applicable):

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	53	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (53 Marks)**

This section has **eight** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(6 marks)**

A system of equations, where b is a real constant, is as follows:

$$x - y + 3z = 11$$
$$x + 2y + 2z = 3$$
$$2x + by + 4z = 8$$

(a) Solve the system when $b = 3$. (4 marks)

Solution
$2(2) - (3)$:
$$y = -2$$
Sub into $(1), (2)$:
$$x + 3z = 9$$ $$x + 2z = 7$$ $$z = 2$$
$$x + 4 = 7$$ $$x = 3$$
Solution:
$x = 3, \quad y = -2, \quad z = 2$
Specific behaviours
✓ indicates correct use of elimination techniques ✓ solves for y ✓ solves for z ✓ solves for x

(b) Interpret the system of equations geometrically when $b = 4$. (2 marks)

Solution
Consider last equation:
$$2x + 4y + 4z = 8 \Rightarrow x + 2y + 2z = 4$$
Compare to second equation:
$$x + 2y + 2z = 3$$
The system consists of two distinct parallel planes cut by a third plane.
Specific behaviours
✓ simplifies equation to enable comparison ✓ identifies parallel planes

Question 2 **(6 marks)**

Polynomial P is defined as $P(z) = z^4 - 4z^3 + 14z^2 - 36z + 45$.

(a) Show that $z - 3i$ is a factor of $P(z)$. (2 marks)

Solution
$P(3i) = (3i)^4 - 4(3i)^3 + 14(3i)^2 - 36(3i) + 45$ $= 81 + 108i - 126 - 108i + 45 \quad \ldots(1)$ $= 0$
Specific behaviours
✓ correctly substitutes $z = 3i$ into $P(z)$ ✓ expands and simplifies terms to obtain (1) and deduces $P(3i) = 0$

(b) Solve $P(z) = 0$, writing solutions in Cartesian form. (4 marks)

Solution
Conjugate of above is another factor: $z + 3i$ Hence $(z + 3i)(z - 3i) = z^2 + 9$ is a factor. $z^4 - 4z^3 + 14z^2 - 36z + 45 = (z^2 + 9)(z^2 - 4z + 5)$ $z^2 - 4z + 5 = 0$ $(z - 2)^2 = -1$ $z = 2 \pm i$ Solutions: $z = \pm 3i, z = 2 \pm i$.
Specific behaviours
✓ indicates conjugate as second factor and obtains quadratic factor ✓ obtains second quadratic factor ✓ solves second quadratic ✓ indicates all solutions

Question 3 **(6 marks)**

Functions f, g and h are defined as

$$f(x) = x + 3, \qquad g(x) = \sqrt{x}, \qquad h(x) = \frac{4}{2 - x}.$$

(a) Determine

(i) $h \circ g \circ f(6)$. (1 mark)

Solution
$h \circ g \circ f(6) = h \circ g(9) = h(3) = -4$
Specific behaviours
✓ correct value

(ii) the defining rule for $h \circ g \circ f(x)$. (1 mark)

Solution
$h \circ g \circ f(x) = h \circ g(x+3)$ $= h\left(\sqrt{x+3}\right)$ $= \dfrac{4}{2 - \sqrt{x+3}}$
Specific behaviours
✓ correct rule

(b) Determine the domain of $h \circ g(x)$. (2 marks)

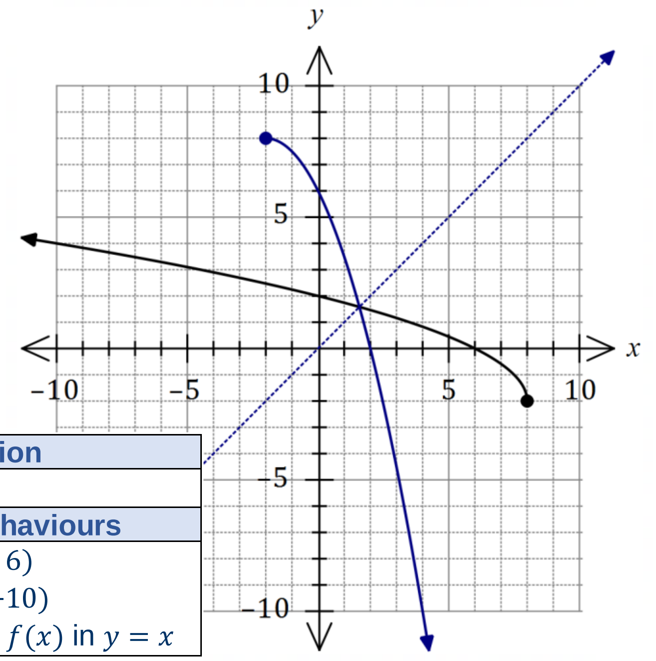
Solution
$x \geq 0$ $2 - \sqrt{x} \neq 0 \Rightarrow x \neq 4$ $D_{h \circ g} = \{x : x \in \mathbb{R}, x \geq 0, x \neq 4\}$
Specific behaviours
✓ states $x \geq 0$ ✓ states $x \neq 4$

(c) Determine the range of $h \circ g(x)$. (2 marks)

Solution
$0 \leq x < 4, \qquad y \geq 2$ $x > 4, \qquad y < 0$ $R_{h \circ g \circ f} = \{y : y \in \mathbb{R}, y \geq 2 \cup y < 0\}$
Specific behaviours
✓ states $y \geq 2$ ✓ states $y < 0$

Question 4 **(6 marks)**

The graph of $y = f(x)$ is shown below.



Solution
See graph
Specific behaviours
✓ $(-2, 8)$ and $(0, 6)$
✓ $(2, 0)$ and $(4, -10)$
✓ symmetry with $f(x)$ in $y = x$

(a) Draw the graph of $y = f^{-1}(x)$ on the same axes. (3 marks)

(b) Given that $f(x) = \sqrt{16 - 2x} - 2$, determine the defining rule for $f^{-1}(x)$. (3 marks)

Solution
$x = \sqrt{16 - 2y} - 2$ $16 - 2y = (x + 2)^2$ $y = 8 - \dfrac{(x+2)^2}{2}$ $D_{f^{-1}} = R_f : y \geq -2$ $f^{-1}(x) = 8 - \dfrac{(x+2)^2}{2}, x \geq -2$
Specific behaviours
✓ removes square root from expression   ✓ obtains correct expression for y in terms of x ✓ writes inverse with domain restriction

Question 5 **(8 marks)**

Points A, B, C and D have position vectors $\overrightarrow{OA} = \begin{pmatrix}-1\\2\\4\end{pmatrix}, \overrightarrow{OB} = \begin{pmatrix}-3\\1\\7\end{pmatrix}, \overrightarrow{OC} = \begin{pmatrix}3\\4\\-2\end{pmatrix}$ and $\overrightarrow{OD} = \begin{pmatrix}0\\0\\4\end{pmatrix}$.

Note that $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ given by $\theta = \cos^{-1}(\hat{\mathbf{u}} \cdot \hat{\mathbf{v}})$.

(a) Determine $|\overrightarrow{AB} \times \overrightarrow{AC}|$ and use the result to explain why A, B and C are collinear. (5 marks)

Solution

$$\overrightarrow{AB} = \begin{pmatrix}-3\\1\\7\end{pmatrix} - \begin{pmatrix}-1\\2\\4\end{pmatrix} = \begin{pmatrix}-2\\-1\\3\end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix}3\\4\\-2\end{pmatrix} - \begin{pmatrix}-1\\2\\4\end{pmatrix} = \begin{pmatrix}4\\2\\-6\end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix}-2\\-1\\3\end{pmatrix} \times \begin{pmatrix}4\\2\\-6\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow |\overrightarrow{AB} \times \overrightarrow{AC}| = 0$$

But $|\overrightarrow{AB} \times \overrightarrow{AC}| = |\overrightarrow{AB}| \cdot |\overrightarrow{AC}| \cdot \sin\theta$ and since $|\overrightarrow{AB}| \neq 0$ and $|\overrightarrow{AC}| \neq 0$ then $\sin\theta = 0 \Rightarrow \theta = 0$.

Hence vectors are parallel as the angle between them is zero, and as A is a point in common, then A, B and C are collinear.

Specific behaviours

- ✓ vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$
- ✓ cross product
- ✓ magnitude of cross product
- ✓ reasons that angle between vectors is zero
- ✓ explains collinearity

(b) Determine the Cartesian equation of the plane containing all four points. (3 marks)

Solution

$$\overrightarrow{AD} = \begin{pmatrix}0\\0\\4\end{pmatrix} - \begin{pmatrix}-1\\2\\4\end{pmatrix} = \begin{pmatrix}1\\-2\\0\end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{pmatrix}-2\\-1\\3\end{pmatrix} \times \begin{pmatrix}1\\-2\\0\end{pmatrix} = \begin{pmatrix}6\\3\\5\end{pmatrix}$$

$$\begin{pmatrix}6\\3\\5\end{pmatrix} \cdot \begin{pmatrix}0\\0\\4\end{pmatrix} = 20$$

$$6x + 3y + 5z = 20$$

Specific behaviours

- ✓ direction vector using D
- ✓ normal using cross product
- ✓ Cartesian equation

Question 6 **(7 marks)**

(a) Determine the complex cube roots of -1 in the form $a + bi$, where $a, b \in \mathbb{R}$. (3 marks)

Solution
$z^3 = -1 = \text{cis}(\pi + 2n\pi)\,, n \in \mathbb{Z}$ $z = \text{cis}\left(\frac{\pi + 2n\pi}{3}\right)$ $z_0 = \text{cis}\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ $z_1 = \text{cis}(\pi) = -1$ $z_2 = \text{cis}\left(-\frac{\pi}{3}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}i$
Specific behaviours
✓ expresses -1 in polar form (or sketch)   ✓ z_0 or z_2 ✓ all roots in required form

(b) Let ω be a complex cube root of unity, $\text{Im}\,\omega \neq 0$, so that $\omega^3 - 1 = 0$.

(i) Show that $(\omega - 1)(\omega^2 + \omega + 1) = \omega^3 - 1$ and hence explain why $\omega^2 + \omega + 1 = 0$. (2 marks)

Solution
$(\omega - 1)(\omega^2 + \omega + 1) = \omega^3 + \omega^2 + \omega - \omega^2 - \omega - 1$ $= \omega^3 - 1$ Since $\omega^3 - 1 = 0$, but $\text{Im}\,\omega \neq 0$ and so $\omega - 1 \neq 0$, then $\omega^2 + \omega + 1 = 0$.
Specific behaviours
✓ fully expands factors   ✓ explanation using factors

(ii) Simplify $(2 + 5\omega)(2 + 5\omega^2)$. (2 marks)

Solution
$(2 + 5\omega)(2 + 5\omega^2) = 4 + 10\omega + 10\omega^2 + 25\omega^3$ $= -6 + 10(1 + \omega + \omega^2) + 25$ $= 19$
Specific behaviours
✓ expands and uses $\omega^3 = 1$ and $\omega^2 + \omega + 1 = 0$ ✓ correct value

Question 7 **(6 marks)**

The equation of line L is

$$\frac{x-2}{2}=\frac{y+1}{-3}=\frac{z-1}{6}.$$

(a) Determine the vector equation of the line in the form $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$. (2 marks)

Solution
$\mathbf{r}=\begin{pmatrix}2\\-1\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\-3\\6\end{pmatrix}$
Specific behaviours
✓ vector $\mathbf{a}$ ✓ vector $\mathbf{b}$

(b) The diameter of sphere S is the segment of line L between $x=2$ and $x=6$. Determine the equation of the sphere. (4 marks)

Solution
Using i-coefficient of line, $x=2\Rightarrow\lambda=0$ and $x=6\Rightarrow\lambda=2$. Hence centre of sphere when $\lambda=1$ at $(4,-4,7)$ Radius $=\left\|\begin{pmatrix}2\\-3\\6\end{pmatrix}\right\|=\sqrt{4+9+36}=7$. Equation: $\left\|\mathbf{r}-\begin{pmatrix}4\\-4\\7\end{pmatrix}\right\|=7$ or $(x-4)^2+(y+4)^2+(z-7)^2=49$
Specific behaviours
✓ identifies location of centre ✓ identifies vector representing radius ✓ correct radius ✓ correct equation in either form

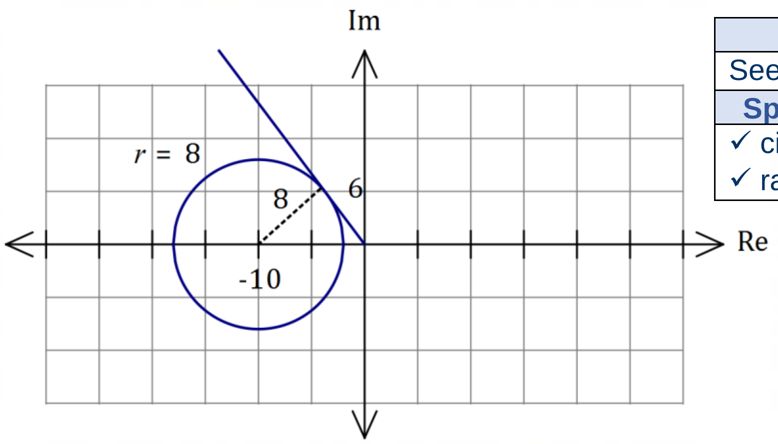
Question 8 **(8 marks)**

The locus L_1 of the complex number $z = x + iy$ has equation $|z - 6| = 2|z + 6|$.

(a) Show that L_1 is a circle with equation $x^2 + y^2 + 20x + 36 = 0$. (2 marks)

Solution
$(x-6)^2 + y^2 = 4((x+6)^2 + y^2)$ $x^2 - 12y + 36 + y^2 = 4x^2 + 48y + 144 + 4y^2$ $3x^2 + 3y^2 + 60y + 108 = 0$ $x^2 + y^2 + 20y + 36 = 0$
Specific behaviours
✓ equates square of magnitudes   ✓ fully expands before simplification

(b) Sketch L_1 on an Argand diagram. (2 marks)



Solution
See graph
Specific behaviours
✓ circle, centre   ✓ radius

Another locus L_2 has equation $w \cdot \bar{z} + \bar{w} \cdot z = 0$, where $w = 4 + 3i$.

(c) Show that L_2 is a tangent to L_1. (4 marks)

Solution
$(4+3i)(x-iy) + (4-3i)(x+iy) = 0$ $4x - 4iy + 3ix + 3y + 4x + 4iy - 3ix + 3y = 0$ $6y = -8x$ $y = -\frac{4x}{3}$ Line intersects circle $(x+10)^2 + \left(-\frac{4x}{3}\right)^2 = 64$ $\frac{25}{9}x^2 + 20x + 36 = 0$ For a tangent, discriminant must be 0 $\Delta = 20^2 - 4\left(\frac{25}{9}\right)(4 \times 9)$ $\Delta = 0$
Specific behaviours
✓ Cartesian equation of L_2 ✓ substitutes line into circle   ✓ derives quadratic   ✓ uses discriminant

End of questions

Supplementary page

Question number: _________